International Journal of Theoretical Physics, Vol. 37, No. 2, 1998

Transformations of Single and Double Hypergeometric Series from the Triple Sum Series for the 9-*j* Coefficient

K. Srinivasa Rao[1] and J. Van der Jeugt[2]

Received December 1, 1997

The Bailey transform for a Saalschützian ${}_4F_3(1)$ and a transformation of a Kampé de Fériet function into a Saalschützian ${}_4F_3(1)$ or its Bailey transform are derived in a novel way in this article: from the simplest known formula for the 9-j coefficient, due to Ališauskas–Jucys–Bandzaitis, which is the triple sum series. It becomes a single, double, or (remains a) triple series when one of the angular momenta is set to zero, due to its inherent lack of symmetry. This then is equated to a ${}_4F_3(1)$ representation of the 6-j coefficient to which the 9-j coefficient reduces when any one of the nine angular momenta is zero.

1. INTRODUCTION

The triple sum series of Ališauskas, Jucys, and Bandzaitis (Ališauskas and Jucys, 1971; Jucys and Bandzaitis, 1977) is the simplest known algebraic form for the 9-j angular momentum recoupling coefficient ubiquitous in atomic, molecular, and nuclear physics. This triple sum series has been shown (Srinivasa Rao and Rajeswari, 1989) to be a particular case of the extremely general triple hypergeometric series studied by Lauricella (1893), Saran (1954), and Srivastava (1964). That this formula does not exhibit any one of the 72 symmetries of the 9-j coefficient was realized while an algorithm was proposed (Srinivasa Rao *et al.*, 1989) for its numerical computation, where an inherent disadvantage was converted into an advantage. In this article, another consequence of this lack of symmetry is exploited.

[1] Department of Applied Mathematics and Computer Science, University of Gent, B-9000 Gent, Belgium. Permanent address: Institute of Mathematical Sciences, C.I.T. Campus, Madras 600 113, India; e-mail: rao@imsc.ernet.in.

[2] Department of Applied Mathematics and Computer Science, University of Gent, B-9000 Gent, Belgium; e-mail: Joris.VanderJeugt@rug.ac.be.

0020-7748/98/0300-0891$15.00/0

There are two hierarchic formulas for the 9-j coefficient. The earliest of these and the most straightforward one expresses the 9-j coefficient as a (double) sum over the projection quantum numbers of the product of six 3-j coefficients. It is this formula which exhibits the 72 symmetries of the 9-j coefficient (up to an overall sign factor) and also reveals how the 9-j coefficient reduces to a 6-j coefficient when any one of its nine angular momenta is set equal to zero. It has been established (Srinivasa Rao and Rajeswari, 1993, for references) that the 6-j coefficient can be expressed in one of seven ways as a terminating Saalschützian ${}_4F_3(1)$.

The triple sum series for the 9-j coefficient, however, does not always reduce to a 6-j coefficient or a single sum series, when any one of the nine angular momenta is set equal to zero, as a consequence of its highly asymmetric nature referred to. Of the nine cases (for setting an angular momentum parameter in the 9-j coefficient to zero), two cases manifestly reduce to single sum series and the other seven cases result in (six) double and (one) triple sum series. An analysis of these special cases is shown here to yield directly either the well-known Bailey transform for a Saalschützian ${}_4F_3(1)$ or a transformation of a certain Kampé de Fériet double series denoted by $F^{0:3;3}_{1:1;1}(1, 1)$ into a Saalschützian ${}_4F_3(1)$ or its Bailey transform.

In Section 2, the basic formulas required in our study are given and in Section 3, the nine cases where a 9-j coefficient reduces to a 6-j coefficient are analyzed and the resulting transformation formulas presented. In Section 4, the results are discussed.

2. FORMULAS FOR THE 9-j COEFFICIENT

The hierarchic formula which can be derived from the recoupling of four angular momenta in two different ways, $\bar{l}_1 + \bar{s}_1 = \bar{j}_1$, $\bar{l}_2 + \bar{s}_2 = \bar{j}_2$, and $\bar{j}_1 + \bar{j}_2 = \bar{J}$, called the jj-coupling, and $\bar{l}_1 + \bar{l}_2 = \bar{L}$, $\bar{s}_1 + \bar{s}_2 = \bar{S}$, and $\bar{L} + \bar{S} = \bar{J}$, called the LS-coupling scheme, expresses the LS-jj transformation coefficient or the 9-j coefficient as a sum over two projection quantum numbers of six 3-j coefficients, each of which corresponds to a coupling of two angular momenta to give a third (Edmonds, 1957). However, the hierarchic formula often used in numerical work by atomic, molecular, and nuclear physicists is

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix} = \sum_x (-1)^{2x}(2x+1)\begin{Bmatrix} a & d & g \\ h & i & x \end{Bmatrix}\begin{Bmatrix} b & e & h \\ d & x & f \end{Bmatrix}\begin{Bmatrix} c & f & i \\ x & a & b \end{Bmatrix} \quad (1)$$

which is a single sum over a product of the three 6-j coefficients. The 6-j

coefficient can be expressed in terms of the ${}_4F_3(1)$'s in two different ways (Srinivasa Rao and Rajeswari, 1993, and references therein):

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = (-1)^{E+1}\Delta(abe)\Delta(cde)\Delta(acf)\Delta(bdf) \times \Gamma(1-E)[\Gamma(1-A, 1-B, 1-C, 1-D, F, G)]^{-1} \times {}_4F_3(A, B, C, D; E, F, G; 1) \tag{2}$$

where

$$A = e - a - b, \qquad B = e - c - d,$$
$$C = f - a - c, \qquad D = f - b - d$$
$$E = -a - b - c - d - 1, \qquad F = e + f - b - c + 1,$$
$$G = e + f - a - d + 1$$

and two other members belonging to this set I of three ${}_4F_3$'s correspond to (2) being written out explicitly for

$$\begin{Bmatrix} e & b & a \\ f & c & d \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} e & a & b \\ f & d & c \end{Bmatrix}$$

Equivalently,

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = (-1)^{A'-2}\Delta(abe)\Delta(cde)\Delta(acf)\Delta(bdf) \times \Gamma(A')[\Gamma(1-B', 1-C', 1-D', E', F', G')]^{-1} \times {}_4F_3(A', B', C', D'; E', F', G'; 1) \tag{3}$$

where

$$A' = a + b + e + 2, \qquad B' = a - c - f,$$
$$C' = b - d - f, \qquad D' = e - c - d$$
$$E' = a + b - c - d + 1, \qquad F' = a + e - d - f + 1,$$
$$G' = b + e - c - f + 1$$

and three other members belonging to this set II of ${}_4F_3(1)$'s correspond to (3) being written out explicitly for

$$\begin{Bmatrix} c & d & e \\ b & a & f \end{Bmatrix}, \quad \begin{Bmatrix} a & c & f \\ d & b & e \end{Bmatrix}, \quad \text{and} \quad \begin{Bmatrix} d & b & f \\ a & c & e \end{Bmatrix}$$

These sets have been shown to be necessary and sufficient to account for the 144 symmetries of the 6-j coefficient. Solving the equations for the parameters of the ${}_4F_3(1)$ on the rhs of (2), taking into account the Saalschützian condition on them, $A + B + C + D + 1 = E + F + G$, for the parameters of the 6-j coefficient, and using these in (3) and equating the rhs of (2) and (3) leads to the "reversal" formula for the ${}_4F_3(1)$. In fact, Srinivasa Rao and Rajeswari (1993) established that the two sets of ${}_4F_3(1)$'s for the 6-j coefficient are related to each other by this reversal formula. In (2) and (3), the function $\Delta(xyz)$ represents

$$\Delta(xyz) = [(-x + y + z)!\,(x - y + z)!\,(x + y - z)!/ (x + y + z + 1)!]^{1/2} \tag{4}$$

which reflects the triangle inequality condition satisfied by three angular momenta and it vanishes if x, y, z do not form a vector triad.

The hierarchic formula where the 9-j coefficient is expressed as a sum over a product of six 3-j coefficients is highly symmetric. It is the one which reveals, via the symmetries of the 3-j coefficient, the 72 symmetries (up to an overall phase factor) of the 9-j coefficient: its invariance to column and row permutations and a reflection about the diagonal of the nine parameters of the symbol on the lhs of (1). It is also the formula which shows explicitly that when any one of the nine angular momenta is set equal to zero, two of the six 3-j coefficients reduce to simple expressions and the remaining four 3-j coefficients can be summed over their projection quantum numbers to yield a 6-j (or Racah) coefficient, defined as the recoupling coefficient of four angular momenta (Edmonds, 1957; Biedenharn and Louck, 1981; Varshalovich *et al.*, 1988). Thus, we have

$$\begin{aligned}
\begin{Bmatrix} 0 & e & e \\ f & d & b \\ f & c & a \end{Bmatrix} &= \begin{Bmatrix} e & 0 & e \\ c & f & a \\ d & f & b \end{Bmatrix} = \begin{Bmatrix} f & f & 0 \\ d & c & a \\ b & a & e \end{Bmatrix} \\
&= \begin{Bmatrix} f & b & d \\ 0 & e & e \\ f & a & c \end{Bmatrix} = \begin{Bmatrix} a & f & c \\ e & 0 & e \\ b & f & d \end{Bmatrix} = \begin{Bmatrix} b & a & e \\ f & f & 0 \\ d & c & e \end{Bmatrix} \\
&= \begin{Bmatrix} e & d & c \\ e & b & a \\ 0 & f & f \end{Bmatrix} = \begin{Bmatrix} c & e & d \\ a & e & b \\ f & 0 & f \end{Bmatrix} = \begin{Bmatrix} a & b & e \\ c & d & e \\ f & f & 0 \end{Bmatrix} \\
&= \frac{(-1)^{b+c+e+f}}{[(2e+1)(2f+1)]^{1/2}} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}
\end{aligned} \tag{5}$$

The simplest known algebraic form for the 9-j coefficient, due to Ališauskas and Jucys (1971), is the triple sum series

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix} = (-1)^{x_5} \frac{(d, a, g)(b, e, h)(i, g, h)}{(d, e, f)(b, a, c)(i, c, f)}$$

$$\times \sum_{x,y,z} \frac{(-1)^{x+y+z}}{x!\, y!\, z!} \frac{(x_1 - x)!\, (x_2 + x)!\, (x_3 + x)!}{(x_4 - x)!\, (x_5 - x)!}$$

$$\times \frac{(y_1 + y)!\, (y_2 + y)!}{(y_3 + y)!\, (y_4 - y)!\, (y_5 - y)!}$$

$$\times \frac{(z_1 - z)!\, (z_2 + z)!}{(z_3 - z)!\, (z_4 - z)!\, (z_5 - z)!}$$

$$\times \frac{(p_1 - y - z)!}{(p_2 + x + y)!\, (p_3 + x + z)!} \tag{6}$$

where

$$\begin{aligned} 0 &\le x \le \min(x_4, x_5) = X \\ 0 &\le y \le \min(y_4, y_5) = Y \\ 0 &\le z \le \min(z_4, z_5) = Z \end{aligned} \tag{7}$$

with

$$\begin{array}{lll} x_1 = 2f, & y_1 = -b + e + h, & z_1 = 2a \\ x_2 = d + e - f, & y_2 = g + h - i, & z_2 = -a + b + c \\ x_3 = c - f + i, & y_3 = 2h + 1, & z_3 = a + d + g + 1 \\ x_4 = -d + e + f, & y_4 = b + e - h, & z_4 = a + d - g \\ x_5 = c + f - i, & y_5 = g - h + i, & z_5 = a - b + c \\ p_1 = a + d - h + i, & p_2 = -b + d - f + h, & p_3 = -a + b - f + i \end{array} \tag{8}$$

and

$$(a, b, c) = \left[\frac{(a - b + c)!\, (a + b - c)!\, (a + b + c + 1)!}{(-a + b + c)!} \right]^{1/2}$$

The lack of symmetry of the triple sum series is due to the nature of the 18 parameters $x_1, x_2, \ldots, p_3$, given in (8), among which nine relations exist.

For the sake of typographic felicity, using the notation for Pochhammer symbols,

$$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \lambda(\lambda + 1) \cdots (\lambda + k - 1), \qquad k \geq 0$$

$$(\lambda)_{-k} = (-1)^k/(1 - \lambda)_k, \qquad k < 0$$

we can rewrite equation (6) as

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix}$$

$$= (-1)^{x_5} \frac{(d, a, g)(b, e, h)(i, g, h)}{(d, e, f)(b, a, c)(i, c, f)}$$

$$\times \frac{\Gamma(1 + x_1, 1 + x_2, 1 + x_3, 1 + y_1, 1 + y_2, 1 + z_1, 1 + z_2, 1 + p_1)}{\Gamma(1 + x_4, 1 + x_5, 1 + y_3, 1 + y_4, 1 + y_5, 1 + z_3, 1 + z_4, 1 + z_5, 1 + p_2, 1 + p_3)}$$

$$\times \sum_{x,y,z} \frac{1}{x!\, y!\, z!} \frac{(1 + x_2)_x(1 + x_3)_x(-x_4)_x(-x_5)_x}{(-x_1)_x} \frac{(1 + y_1)_y(1 + y_2)_y(-y_4)_y(-y_5)_y}{(1 + y_3)_y}$$

$$\times \frac{(1 + z_2)_z(-z_3)_z(-z_4)_z(-z_5)_z}{(-z_1)_z} \frac{1}{(-p_1)_{y+z}(1 + p_2)_{x+y}(1 + p_3)_{x+z}} \tag{9}$$

where

$$\Gamma(x, y, \ldots) = \Gamma(x)\Gamma(y) \cdots \tag{10}$$

Here (6) and (9) are equivalent ways of writing the triple sum series and we give both these here, since it is advantageous to choose one or the other as the starting point to obtain our results.

3. TRANSFORMATIONS

It is to be noted, from (7), that when we set $a, b, c, \ldots, i = 0$, in succession, for $c = 0$ and $e = 0$, $X = 0 = Z$ and $X = 0 = Y$, respectively, so that (6) or (9) reduces to a single sum over y or z. The seven other cases, except $h = 0$, reduce (9) to a double sum series, since only one of X, Y, Z becomes 0. For $h = 0$, which occurs in both y_4 and y_5, though $b = e$ and $g = i$, neither y_4 nor y_5 becomes 0, and so the triple sum remains a triple sum except for its reduction to a function of six angular momenta. Obviously, the number of independent angular momenta reduces to six in all the nine cases.

First, we consider the cases $c = 0$ and $e = 0$, which reduce (9) to a single sum series in either y or z, respectively. For $c = 0$, we have

$$\begin{Bmatrix} a & a & 0 \\ d & e & f \\ g & h & f \end{Bmatrix} = \frac{(d, a, g)(a, e, h)(f, g, h)}{(d, e, f)(a, a, 0)(f, 0, f)}$$

$$\times \frac{\Gamma(1 + x_1, 1 + x_2, 1 + y_1, 1 + y_2, 1 + z_1, 1 + p_1)}{\Gamma(1 + x_4, 1 + y_3, 1 + y_4, 1 + y_5, 1 + z_3, 1 + z_4, 1 + p_2)}$$

$$\times {}_4F_3\left(\begin{matrix}1+y_1,\ 1+y_2,\ -y_4,\ -y_5\\ -p_1,\ 1+p_2,\ 1+y_3\end{matrix};1\right) \tag{11}$$

where

$$\begin{aligned}
&x_1 = 2f && y_1 = -a+e+h && z_1 = 2a\\
&x_2 = d+e-f && y_2 = g+h-f && z_3 = a+d+g+1\\
&x_4 = -d+e+f && y_3 = 2h+1 && z_4 = a+d-g\\
&p_1 = a+d-h+f, && y_4 = a+e-h &&\\
& && y_5 = g-h+f &&\\
& && p_2 = -a+d-f+h, &&
\end{aligned} \tag{12}$$

From (5) we know that

$$\begin{Bmatrix} a & a & 0\\ d & e & f\\ g & h & f\end{Bmatrix} = \frac{(-1)^{a+e+f+g}}{[(2a+1)(2f+1)]^{1/2}}\begin{Bmatrix} h & g & f\\ d & e & a\end{Bmatrix} \tag{13}$$

Using (2), and a symmetry of the 6-j coefficient, we rewrite (13) as

$$\begin{aligned}
\begin{Bmatrix} a & a & 0\\ d & e & f\\ g & h & f\end{Bmatrix} &= \frac{(-1)^{a+e+f+g}(-1)^{E+1}}{[(2a+1)(2f+1)]^{1/2}}\,\Delta(aeh)\Delta(fgh)\Delta(agd)\Delta(fed)\\
&\quad\times \Gamma(1-E)[\Gamma(1-A,1-B,1-C,1-D,F,G)]^{-1}\\
&\quad\times {}_4F_3\left(\begin{matrix}A,\ B,\ C,\ D\\ E\ F,\ G\end{matrix};1\right)
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
A &= h-a-e, \qquad B = h-f-g,\\
C &= d-a-g, \qquad D = d-e-f\\
E &= -a-e-f-g-1, \qquad F = d+h-e-g+1,\\
G &= d+h-a-f+1
\end{aligned} \tag{15}$$

and ${}_4F_3(1)$ is a Saalschützian. Solving (15) for a, d, e, f, g, and h, we rewrite (12) in terms of these, and after simplifications, we get, on equating (14) and (11),

$$
{}_4F_3\left(\begin{matrix} A, B, C, D \\ E, F, G \end{matrix}; 1\right)
= \frac{\Gamma(E+F-A-B-D, F, E+F-A-B-C, F-C-D)}{\Gamma(F-C, F-D, 1-G, E+F-A-B)}
\times {}_4F_3\left(\begin{matrix} E-A, E-B, C, D \\ E, E+F-A-B, E+G-A-B \end{matrix}; 1\right) \tag{16}
$$

which is exactly the Bailey transformation for the Saalschützian ${}_4F_3(1)$ [cf. Slater, 1966, p. 64, equation (2.4.1.7)].

The case $e = 0$ proceeds along the same lines and it also yields (16), the Bailey transform for a Saalschützian ${}_4F_3(1)$.

For $a = 0$, the triple sum in (9) reduces to a double sum over x and y:

$$
\begin{Bmatrix} 0 & b & b \\ d & e & f \\ d & h & i \end{Bmatrix} = (-1)^{x_5} \frac{(d, 0, d)(b, e, h)(i, d, h)}{(d, e, f)(b, 0, b)(i, b, f)}
\times \frac{\Gamma(1+x_1, 1+x_2, 1+y_1, 1+y_2, 1+z_2)}{\Gamma(1+x_4, 1+x_5, 1+y_3, 1+y_4, 1+z_3, 1+p_2)}
\times F^{0:3;3}_{1:1;1}\left(\begin{matrix} - \\ 1+p_2 \end{matrix}; \begin{matrix} 1+x_2, -x_4, -x_5 \\ -x_1 \end{matrix}; \begin{matrix} 1+y_1, 1+y_2, -y_4 \\ 1+y_3 \end{matrix}; 1, 1\right) \tag{17}
$$

where

$$
\begin{aligned}
x_1 &= 2f, & y_1 &= -b+e+h, & z_2 &= 2b \\
x_2 &= d+e-f, & y_2 &= d+h-i, & z_3 &= 2d+1 \\
x_4 &= -d+e+f, & y_3 &= 2h+1, & & \\
x_5 &= b+f-i, & y_4 &= b+e-h, & & \\
& & p_2 &= -b+d-f+h, & &
\end{aligned} \tag{18}
$$

and we have used the definition of the double hypergeometric function (cf. Van der Jeugt, *et al.*, 1994):

$$
F^{A:B;B'}_{C:D;D'}\left(\begin{matrix} (a) \\ (c) \end{matrix} : \begin{matrix} (b) \\ (d) \end{matrix}; \begin{matrix} (b') \\ (d') \end{matrix}; x, y\right)
= \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{A} (a_j)_{m+n}}{\prod_{j=1}^{C} (c_j)_{m+n}} \frac{\prod_{j=1}^{B} (b_j)_m}{\prod_{j=1}^{D} (d_j)_m} \frac{\prod_{j=1}^{B'} (b'_j)_n}{\prod_{j=1}^{D'} (d'_j)_n} \frac{x^m y^n}{m! n!} \tag{19}
$$

which is a special case of a very general function defined by Srivastava and

Daoust (1969). We note that in (17) though $-x_1$ appears as a denominator parameter, the zero due to this will occur only after the zero in the numerator occurs due to $-x_4$, $-x_5$. Similar situations arise later on in (31) and (35) also.

From (5) and (3), we have

$$\begin{Bmatrix} 0 & b & b \\ d & e & f \\ d & h & i \end{Bmatrix} = \frac{(-1)^{b+d+f+h}}{[(2b+1)(2d+1)]^{1/2}} \begin{Bmatrix} i & f & b \\ e & h & d \end{Bmatrix}$$

$$= \frac{(-1)^{b+d+f+h}}{[(2b+1)(2d+1)]^{1/2}} (-1)^{A_1-2} \Delta(ifb)\Delta(beh)\Delta(dhi)\Delta(efd)$$

$$\times \Gamma(A_1)[\Gamma(1-B_1, 1-C_1, 1-D_1, E_1, F_1, G_1)]^{-1} {}_4F_3\left(\begin{matrix} A_1, B_1, C_1, D_1 \\ E_1, F_1, G_1 \end{matrix}; 1\right) \tag{20}$$

where

$$\begin{gathered} A_1 = d + h + i + 2, \qquad B_1 = h - e - b, \\ C_1 = i - f - b, \qquad D_1 = d - e - f \\ E_1 = h + i - e - f + 1, \qquad F_1 = d + i - b - e + 1, \\ G_1 = d + h - b - f + 1 \end{gathered} \tag{21}$$

We now introduce a set of parameters

$$\begin{aligned} A &= 1 + x_2, \qquad B = -x_4, \qquad C = -x_5, \\ D &= 1 + p_2, \qquad E = -x_1, \qquad E' = 1 + y_3 \end{aligned} \tag{22}$$

in terms of which (17) and (20) are rewritten, and on equating the two, after simplifications, we get

$$F^{0:3;3}_{1:1;1}\left(\begin{matrix} - \\ D \end{matrix} : \begin{matrix} A, B, C \\ E \end{matrix} ; \begin{matrix} D-A, D-B, D-C \\ E' \end{matrix} ; 1, 1\right)$$

$$= \frac{\Gamma(A+B+C-2D+E', 1+B-E, 1+C-E, E')}{\Gamma(1-E, A-D+E', B+C-D+E', 1+B+C-E)}$$

$$\times {}_4F_3\left(\begin{matrix} A+B+C-D-E+E', D-A, B, C \\ D, B+C-D+E', 1+B+C-E \end{matrix}; 1\right) \tag{23}$$

which is a transformation of a Kampé de Fériet function into a Saalschützian ${}_4F_3(1)$.

When b is set equal to zero, the triple sum (6) reduces to a double sum over x and z. However, since $z_2 = 0$ and $z_5 = z_1$, the z-sum is

$$\sum_z \frac{(-1)^z (p_1 - z)!}{(z_3 - z)!\,(z_4 - z)!\,(p_3 + x + z)!} \tag{24}$$

Changing the summation index $z \to z_3 - z$, we have for the z-sum

$$(-1)^{z_3} \sum_z \frac{(-1)^z (p_1 - z_3 + z)!}{z!\,(z_4 - z_3 + z)!\,(p_3 + z_3 + x - z)!}$$

$$= \frac{(-1)^{z_3}(p_1 - z_3)!}{(z_4 - z_3)!\,(p_3 + z_3 + x)!}\, {}_2F_1\left(\begin{matrix} 1 + p_1 - z_3, -p_3 - z_3 - x \\ 1 + z_4 - z_3 \end{matrix}; 1\right) \tag{25}$$

which can be summed using the Gauss summation theorem to give for the rhs of (25)

$$\frac{(-1)^{z_3}(p_1 - z_3)!\,\Gamma(z_3 + z_4 - p_1 + p_3 + x)}{(p_3 + z_3 + x)!\,\Gamma(z_4 - p_1, 1 + z_4 + p_3 + x)} \tag{26}$$

where $z_3 = a + d + g + 1$, $z_4 = a + d - g$, $p_1 = a + d - e + i$, and $p_3 = -a - f + i$. This reduces the double sum to a single sum over x and the resulting ${}_4F_3(1)$, when equated to the corresponding ${}_4F_3(1)$ from (5), can be shown to be nothing but the Bailey transform for terminating ${}_4F_3(1)$'s given by (16).

The cases $d = 0$ and $i = 0$ are similar to the case $b = 0$ in that they result in double sums (over x and y or x and z, respectively) and in both cases, the ${}_2F_1$ Gauss summation can be used to perform the x-sum. The resulting single sum series are ${}_4F_3(1)$'s which are Saalschützian, and on being equated to the corresponding appropriate ${}_4F_3(1)$'s from (5), yield the Bailey transformation (16).

The case $f = 0$ results from (9) in a genuine double sum over y and z, and when it is equated to the appropriate ${}_4F_3(1)$ form for the 6-j coefficient in (5), we obtain the following Kampé de Fériet transformation:

$$F^{0:3;3}_{1:1;1}\left(\begin{matrix} - \\ D \end{matrix} : \begin{matrix} A, B, C \\ E \end{matrix} ; \begin{matrix} D - A, D - B, D - C \\ E' \end{matrix} ; 1, 1\right)$$

$$= (-1)^{B+C-D} \frac{\Gamma(A + B + C - 2D + E', 1 + D - B - E', 1 + D - C - E')}{\Gamma(A - D + E', 1 + D - E')}$$

$$\times \frac{\Gamma(E - B - C, E, 1 + B - D, 1 + C - D)}{\Gamma(1 - D, 1 - E', E - B, E - C, 1 + B + C - D)}$$

$$\times\, {}_4F_3\left(\begin{matrix} 1 - A + D - E', 1 + A + B + C - D - E, B, C \\ 1 + B + C - E, 1 + B + C - D, 1 + D - E' \end{matrix}; 1\right) \tag{27}$$

Examination of the numerator and denominator parameters of the ${}_4F_3(1)$'s in (23) and (27) shows that two of the numerator parameters B and C, and a denominator parameter $1 + B + C - E$, occur in both. It can be shown by using the property of the Pochhammer symbols (or gamma functions),

$$(1 - z - n)_n = (-1)^n (z)_n \tag{28}$$

that the two ${}_4F_3(1)$'s are related to each other by the Bailey transformation (16). Thus, the transformations (23) and (27) are related to each other through the Bailey transformation for the ${}_4F_3(1)$'s.

For $g = 0$, we get a double sum series over x and z. However, in the parameters of the z-sum,

$$z_1 = 2a, \qquad z_2 = -a + b + c, \qquad z_3 = 2a + 1 = z_1 + 1 \tag{29}$$
$$z_5 = a - b + c, \qquad p_3 = -a + b - f + h$$

a numerator and denominator parameter differ by 1, so that from (9)

$$\sum_z \frac{(1 + z_2)_z(-z_3)_z(-z_5)_z}{z!\,(-z_1)_z(1 + p_3)_{x+z}} = \frac{1}{(1 + p_3)_x}\, {}_3F_2\left(\begin{matrix} a', b', b_1 + m_1 \\ b' + 1, b_1 \end{matrix}; 1\right) \tag{30}$$

where $a' = -z_5$, $b' = -z_1 - 1$, $b_1 = 1 + p_3 + x$, and $m_1 = z_2 - p_3 - x$. The ${}_3F_2$ is summable using Minton's (1970) theorem and results in

$$\frac{1}{(1 + p_3)_x} \frac{\Gamma(1 - a', 1 + b')}{\Gamma(1 - a' + b')} \frac{(b_1 - b')_{m_1}}{(b_1)_{m_1}}$$

which can be simplified to

$$\frac{\Gamma(-z_1, 1 + z_5, 1 + p_3)}{\Gamma(1 + z_2, -z_1 + z_5, 2 + p_3 + z_1)} \frac{1}{(2 + p_3 + z_1)_x}$$

The Pochhammer x factor from this Minton sum reduces the double sum and results in the Saalschützian:

$${}_4F_3\left(\begin{matrix} 1 + x_2, 1 + x_3, -x_4, -x_5 \\ -x_1, 1 + p_2, 2 + p_3 + z_1 \end{matrix}; 1\right) \tag{31}$$

which, on being equated to the corresponding ${}_4F_3(1)$ from (5) [and (2)], gives us again the Bailey transformation (16).

Finally, the case $h = 0$ is the one in which the triple sum remains as a triple sum, since X, Y, and Z are nonzero. However, the set of parameters (8) in this case is

$$\begin{array}{lll} x_1 = 2f, & y_1 = 0, & z_1 = 2a \\ x_2 = b + d - f, & y_2 = 0, & z_2 = -a + b + c \\ x_3 = c - f + g, & y_3 = 1, & z_3 = a + d + g + 1 \end{array} \tag{32}$$

$$x_4 = b - d + f, \qquad y_4 = 2b, \qquad z_4 = a + d - g$$
$$x_5 = c + f - g, \qquad y_5 = 2g, \qquad z_5 = a - b + c$$
$$p_1 = a + d + g, \qquad p_2 = -b + d - f, \qquad p_3 = -a + b - f + g$$

Since $y_1 = y_2 = 0$, $y_3 = 1$, and $z_3 = p_1 + 1$, the sum over y in (6) becomes

$$\sum_y \frac{(-1)^y y!\,(p_1 - z - y)!}{(y+1)!\,(y_4 - y)!\,(y_5 - y)!\,(p_2 + x + y)!} \tag{33}$$

Changing the summation index from $y \to y_4 - y$ and rearranging this sum results in a ${}_3F_2(1)$ of the form given in (30), but with

$$\begin{aligned} a' &= -p_2 - y_4 - x, \\ b' &= -y_4 - 1, \\ b_1 &= 1 - y_4 + y_5, \\ m_1 &= p_1 - y_5 - z \end{aligned}$$

Using the Minton theorem to sum this ${}_3F_2(1)$, after simplifications, the y-sum (33) becomes

$$\frac{(-1)^{x_2+y_4+2x+z}\Gamma(2+p_1, 1-p_2)}{\Gamma(2+y_4, 2+y_5)(-1-p_1)_z(p_2)_x} \tag{34}$$

The parameters (32) reveal $p_2 = -x_4$ and $1 + p_1 = z_3$. Due to these relations, when we use (34) for (33) in (6), we get for the $h = 0$ case

$$\begin{Bmatrix} a & b & c \\ d & b & f \\ g & 0 & g \end{Bmatrix} = \frac{(-1)^{x_2+y_4}}{\Gamma(2+y_4, 2+y_5)} \frac{(d,a,g)(b,b,0)(g,g,0)}{(d,b,f)(b,a,c)(g,c,f)}$$
$$\times \sum_{x,z} \frac{(-1)^z}{x!\,z!} \frac{(x_1 - x)!\,(x_2 + x)!\,(x_3 + x)!\,(z_1 - z)!\,(z_2 + z)!}{(x_5 - x)!\,(p_3 + x + z)!\,(z_4 - z)!\,(z_5 - z)!}$$

which can be rearranged into a Kampé de Fériet function:

$$\begin{Bmatrix} a & b & c \\ d & b & f \\ g & 0 & g \end{Bmatrix} = \frac{(-1)^{x_2+x_4+x_5}}{\Gamma(2+y_4, 2+y_5)} \frac{(d,a,g)(b,b,0)(g,g,0)}{(d,b,f)(b,a,c)(g,c,f)} \frac{x_1!\,x_2!\,x_3!\,z_1!\,z_2!}{x_5!\,z_4!\,z_5!}$$
$$\times F^{0:3;3}_{1:1;1}\left(\begin{matrix} - \\ 1 + p_3 \end{matrix} : \begin{matrix} 1 + x_2, 1 + x_3, -x_5 \\ -x_1 \end{matrix} ; \begin{matrix} 1 + z_2, -z_4, -z_5 \\ -z_1 \end{matrix} ; 1, 1 \right) \tag{35}$$

Thus, this case $h = 0$ also reduces to a double sum series and when this is equated to the ${}_4F_3(1)$ which corresponds to the 6-j coefficient in (5), we get the transformation (23) after we identify

$$A = 1 - a + b + c, \qquad B = -a + b - c, \qquad C = -a - d + g$$
$$D = 1 - a + b - f + g, \qquad E = 2a, \qquad E' = -2f$$

4. DISCUSSION OF RESULTS

The realization that setting any one of the nine angular momenta in the 9-j coefficient reduces always to a 6-j coefficient, through the hierarchic formula for the 9-j, but that the highly asymmetric triple-sum-series form becomes a single series in two cases, a double series in six cases, and remains a triple series in one case, led us to the study of these nine cases in this article. We have shown that in two cases, we obtain the Bailey transformation of Saalschützian ${}_4F_3(1)$'s and in all the other seven cases, the transformation of a certain Kampé de Fériet double series into a Saalschützian ${}_4F_3(1)$ or its Bailey transform. Since the ${}_4F_3(1)$'s involved are Saalschützian, the transformations involved are for functions of six independent parameters.

The Bailey transformation of a terminating ${}_4F_3(1)$ is a well-known result and it is derived here from the 9-j recoupling angular momentum coefficient (or from the Racah–Wigner algebra for quantum theory of angular momentum). Using the Jucys mirror symmetry, $j \to -j - 1$, Varshalovich *et al.* (1988; see also Vilenkin and Klimyk, 1991) have shown that the Bailey transformation for the Saalschützian ${}_4F_3(1)$ can be derived from the ${}_4F_3(1)$ representations for the 6-j coefficient. On the other hand, we have shown (Srinivasa Rao *et al.*, 1975) that the use of the Bailey transformation on a ${}_4F_3(1)$ representation of the 6-j coefficient belonging to set I (2) or set II (3) of ${}_4F_3(1)$'s results *at best* in the mathematical Jucys symmetry and otherwise leads to a symmetry like that obtained by Minton (1970) for the 6-j coefficient.

The Kampé de Fériet double series transformation into a Saalschützian ${}_4F_3(1)$, or its Bailey transform, can be shown to be not new. It is, in fact, the terminating form of a formula given by Karlsson (1994) in his proof of one of our double hypergeometric summation formulas [cf. Van der Jeugt *et al.*, 1994, equation (33)]. The formula in question is [Karlsson, 1994, equation (5)]

$$F^{0:3;3}_{1:1;1}\left(\begin{matrix} -: & a, b, c & ; & a', b', c' \\ d: & e & ; & e' \end{matrix} ; 1, 1\right)$$
$$= \frac{\Gamma(d, e', a' + b' - d, d + e' - a' - b' - c')}{\Gamma(a', b', e' - c', d + e' - a' - b')}$$
$$\times\, {}_4F_3\left(\begin{matrix} a, b, e - c, d + e' - a' - b' - c' \\ e, 1 + d - a' - b', d + e' - a' - b' \end{matrix} ; 1\right)$$

$$+ \frac{\Gamma(d, e, a+b-d, d+e-a-b-c)}{\Gamma(a, b, e-c, d+e-a-b)} \times {}_4F_3\left(\begin{matrix} a', b', e'-c', d+e-a-b-c \\ e', 1+d-a-b, d+e-a-b \end{matrix}; 1\right)$$

which holds for $a' = d - a$ and $b' - d = b$, under the condition $\Re(1 - d + c + c') > 0$, in addition to the conditions $\Re(d + e - a - b - c) > 0$ and $\Re(d + e' - a' - b' - c') > 0$. It is also one of a set of transformation formulas of Kampé de Fériet series given by Pitre and Van der Jeugt (1996). Consider now in the above formula the extra relation $c' = d - c$, and interchange a and c ($a \leftrightarrow c$) in the resulting formula. In it, if b or c is a negative integer, then the second term on the rhs vanishes, so that we have

$$F^{0:3;3}_{1:1;1}\left(\begin{matrix} -: a, b, c; d-a, d-b, d-c \\ d: \quad e \quad ; \quad e' \end{matrix}; 1, 1\right) = \frac{\Gamma(d, e', d-b-c, a+b+c-2d+e')}{\Gamma(d-b, d-c, e'+a-d, b+c-d+e')} \times {}_4F_3\left(\begin{matrix} b, c, e-a, a+b+c-2d+e' \\ b+c-d+e', e, 1+b+c-d \end{matrix}; 1\right)$$

Bailey transform the ${}_4F_3(1)$ on the rhs and use the property of the Pochhammer symbols, $(\alpha)_{-n} = (-1)^n/(1 - \alpha)_n$, to obtain (23) (with a, b, c, d, e, e' replacing A, B, C, D, E, E').

It has to be noted that the results presented here made use of that ${}_4F_3(1)$ form for the 6-j coefficient, belonging to set I or set II, which has the larger number of parameters expressible directly in terms of the parameters (8) of the triple sum series. This is achieved by inspection and with the help of Mathematica. In fine, though the results obtained are not net, the methodology to obtain these transformations is novel and it is a direct consequence of the triple sum series of Ališauskas, Jucys, and Bandzaitis and the fundamental property of the 9-j coefficient that it reduces to a 6-j coefficient when any one of its angular momenta is set to zero.

ACKNOWLEDGMENTS

K.S.R. wishes to thank Prof. G. Vanden Berghe for the excellent hospitality at the University of Ghent and for his interest in the work. This work was partly supported by the E.C. (contract No. CI1*-CT92-0101). J.V.d.J. is a Research Associate of the Fund for Scientific Research, Flanders, Belgium.

REFERENCES

Ališuskas, S. J., and Jucys, A. P. (1971). *Journal of Mathematical Physics,* **12**, 594.

Biedenharn, L. C., and Louck, J. D. (1981). *The Racah–Wigner Algebra in Quantum Theory,* Encyclopedia of Mathematics and Its Applications, Vol. 9, Academic Press, New York.

Edmonds, A. R. (1957). *Angular Momentum in Quantum Mechanics,* Princeton University Press, Princeton, New Jersey.

Jucys, A. P., and Bandzaitis, A. A. (1977). *Angular Momentum in Quantum Physics,* Mokslas, Vilnius, Lithuania.

Karlsson, P. W. (1994). *Journal of Physics A,* **27**, 6943.

Lauricella, G. (1893). *Rendiconti del Circolo Matematico di Palermo,* **1**, 111.

Minton, B. M. (1970). *Journal of Mathematical Physics,* **11**, 1375.

Pitre, S. N., and Van der Jeugt, J. (1996). *Journal of Mathematical Analysis and Applications,* **202**, 121.

Saran, S. (1954). *Ganita,* **5**, 77; Addendum, *ibid.* (1956) **7**, 65.

Slater, L. J. (1966). *Generalized Hypergeometric Functions,* Cambridge University Press, Cambridge.

Srinivasa Rao, K., and Rajeswari, V. (1989). *Journal of Mathematical Physics,* **30**, 1016.

Srinivasa Rao, K., and Rajeswari, V. (1993). *Quantum Theory of Angular Momentum: Selected Topics,* Springer, Berlin, and Narosa, Delhi.

Srinivasa Rao, K., and Venkatesh, K. (1977). In *Group Theoretical Methods in Physics,* Proceedings of the Vth International Colloquium, University of Montreal, Sharp, R. T., and Kolman, B., eds., Academic Press, New York, p. 649.

Srinivasa Rao, K., Rajeswari, V., and Chiu, C. B. (1989). *Computer Physics Communications,* **56**, 231.

Srinivasa Rao, K., Santhanam, T. S., and Venkatesh, K. (1975). *Journal of Mathematical Physics,* **16**, 1528.

Srivastava, H. M. (1964). *Ganita,* **15**, 97.

Srivastava, H. M., and Daoust, M. C. (1969). *Proceedings of the Nederlandse Akademie der Wetenschappen A,* **72**, 449.

Van der Jeugt, J., Pitre, S. N., and Srinivasa Rao, K. (1994). *Journal of Physics A,* **27**, 5251.

Varshalovich, D. A., Moskalev, A. N., and Khersonskii, V. K. (1975). *Quantum Theory of Angular Momentum,* Nauka, Leningrad [in Russian; English translation, 1988 World Scientific, Singapore].

Vilenkin, N. Ja., and Klimyk, A. U. (1991). *Representations of Lie Groups and Special Functions,* Vol. 1, Kluwer, Dordrecht.